Workshop 1

Characteristic Functions and Regime Switching Models

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CCFEA Workshop on Computational Finance

The characteristic function

For any random variable X we define the characteristic function (c.f.) as

$$\phi(\theta) = \mathsf{E}\exp\{\mathtt{i}\theta X\} = \int_{\mathbb{R}} \exp\{\mathtt{i}\theta x\} f(x)\mathtt{d}x$$

The c.f. is has three important features

- the moments are retrieved with differentiation
- the probability density function (p.d.f.) is retrieved with integration
- the c.f. uniquely determines the distribution function

The moments

Observe that if we take the m-th derivative of the c.f. we obtain

$$\phi^{(m)}(\theta) = \frac{\mathsf{d}}{\mathsf{d}\theta}\mathsf{E}\exp\{\mathtt{i}\theta X\} = \mathsf{E}(\mathtt{i}X)^m \exp\{\mathtt{i}\theta X\}$$

Therefore, one can see that the uncentered moments of X can be retrieved as the derivatives evaluated at zero, since

$$\mathsf{E}X^m = \frac{\phi^{(m)}(0)}{\mathtt{i}^m}$$

The density function

The Fourier inversion theorem can provide us with the p.d.f.

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\{-\mathtt{i}\theta x\}\phi(\theta)\mathtt{d}\theta$$

De Moivre's formula gives

$$\exp\{\mathtt{i}\theta\} = \cos\theta + \mathtt{i}\sin\theta$$

Since $\phi(\theta) = \overline{\phi(-\theta)}$, we can simplify the expression

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \{\mathscr{R}(\theta)\cos\theta x + \mathscr{I}(\theta)\sin\theta x\}\mathtt{d}\theta$$

The cumulative density function

The cumulative density function (c.d.f.) $F(x) = \mathsf{P}\{X < x\}$ can be obtained by integration too

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_{\mathbb{R}_+} \mathsf{Re}\left(\frac{\exp\{-\mathtt{i}\theta x\}\phi(\theta)}{\mathtt{i}\theta}\right)\mathtt{d}\theta$$

or alternatively as

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\int_{\mathbb{R}_+} \frac{\mathscr{R}(\theta)\sin\theta x - \mathscr{I}(\theta)\cos\theta x}{\theta}\mathtt{d}\theta$$

Option pricing

The c.f. of the log-asset price can be used to compute European option prices

$$C(k) = S(0)\Pi_1 + K\exp(-rT)\Pi_2$$

with k the log-strike, and the "delta" Π_1 and the exercise probability Π_2 given by

- $\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_{\mathbb{R}+} \mathrm{Re}\left(\frac{\exp\{-\mathtt{i}uk\}\phi(u-\mathtt{i})}{\mathtt{i}u\phi(-\mathtt{i})}\right)\mathrm{d}u$
- $\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_{\mathbb{R}+} \mathrm{Re}\left(\frac{\exp\{-\mathtt{i}uk\}\phi(u)}{\mathtt{i}u}\right)\mathrm{d}u$

Fast Fourier Transforms

The Fast Fourier Transform (FFT) is an efficient algorithm that computes the sum

$$y(k) = \sum_{j=1}^{N} \exp\{-\mathtt{i}\frac{2\pi}{N}(j-1)(k-1)\}x(j)$$

for $k = 1, \dots, N$, given a vector of values $\{x(j)\}_{j=1}^{N}$

- The value N is a power of 2 for added efficiency. The procedure reduces the number of computations from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$

FFT and option pricing

- The FFT algorithm can be used to compute the various integrals.
- Care has to be taken because the integrals can be highly oscillatory. Carr and Madan (1999) discuss this issue.
- There is a potentially severe tradeoff in the FFT procedure between the domain of the c.f. and the domain of the result. The newly developed fractional FFT can offer some alternatives.

Practical FFT

- The expressions for Π_1 and Π_2 are apparently singular at $u = 0$. To circumvent that one introduces the modified option price

$$c(k) = \exp(\alpha k)C(k)$$

- The dumping parameter α has to be introduced. The c.f. of the modified option price $c(k)$ is given by

$$\psi(u) = \frac{\exp\{-rT\}\phi(u - \mathtt{i}(\alpha + 1))}{\alpha^2 + \alpha - u^2 + \mathtt{i}(2\alpha + 1)u}$$

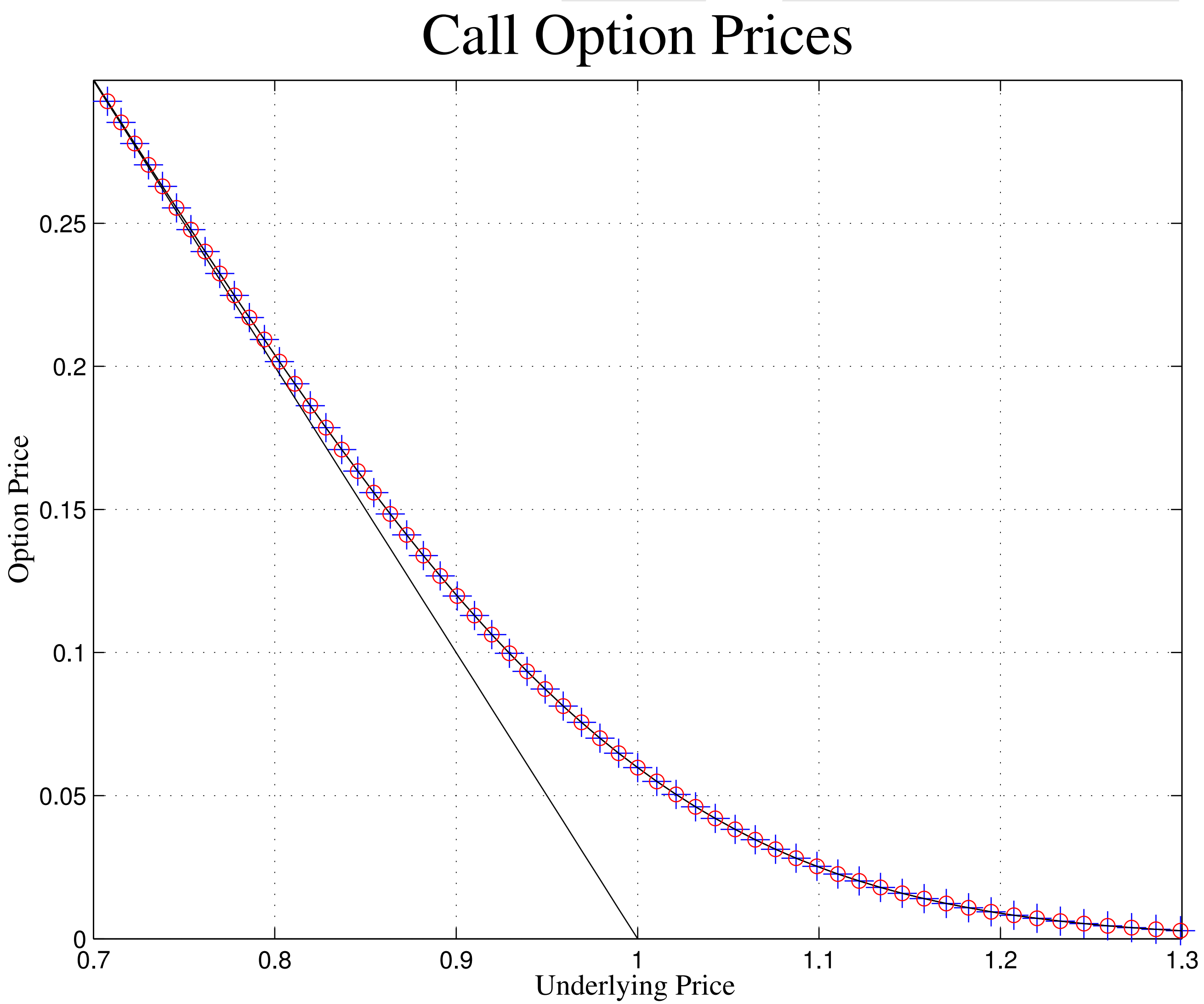
Practical FFT cont'

- Then the option price can get in a form that can be computed using FFT

$$C(k_i) \cong \frac{\exp\{-\alpha k_i\}}{\pi} \sum_{j=1}^{N} \exp\{-\mathtt{i}\frac{2\pi}{N}(j-1)(i-1)\} \exp\{\mathtt{i} b u_j\}\psi(u_j)\eta$$

- Where the c.f. grid extends across $(j-1)\eta$ for $j = 1, \ldots, N$, and the log-price grid extends across $-\frac{\pi}{\eta} + (i-1)\frac{2\pi}{N\eta}$ for $j = 1, \ldots, N$.

The Black-Scholes prices



Markov chains

- A Markov chain in discrete time is defined by the probabilities

$$\mathsf{Pr}\{x_{t+1} = j | x_t = i\} = p_{ji}$$

- A Markov chain in continuous time is defined by the rates (infinitesimal probabilities)

$$\mathsf{Pr}\{x_{t+\delta} = j | x_t = i\} = q_{ji}\delta + o(\delta)$$

- The N states $1, 2, \ldots, N$ define the *state of the world*, or the state of the economy.

Regime Switching Models

- A regime switching model is one where the model parameters change over time, following the unobserved state of the world.
- Therefore only conditional (on the state) inference can be readily made. The challenges are:
  1. estimate the parameters
  2. filter out the unobserved state of the world, based on the observations (filtering problem)
  3. retrieve past states of the world (smoothing problem)
  4. forecast future states of the world

Estimation

The estimation procedure is discussed in detail in Hamilton (1994). In general the steps follow the Kalman filter. At time t we start from the probabilities $\xi_{t-1|t-1}(i) = \mathsf{Pr}\{x_{t-1} = i|\mathscr{F}_{t-1}\}$ and the data y_t.

1. State forecast: $\xi_{t|t-1} = \mathbf{P}\xi_{t-1|t-1}$
2. Density Weighting: $\zeta_t(i) = f(y_t|x_t = i)\xi_{t|t-1}(i)$
3. State filtering: $\xi_{t|t}(i) = \frac{\zeta_t(i)}{\sum_j \zeta_t(j)}$

A byproduct is the log-likelihood which is

$$\sum_t \log \sum_j \zeta_t(j)$$

An example

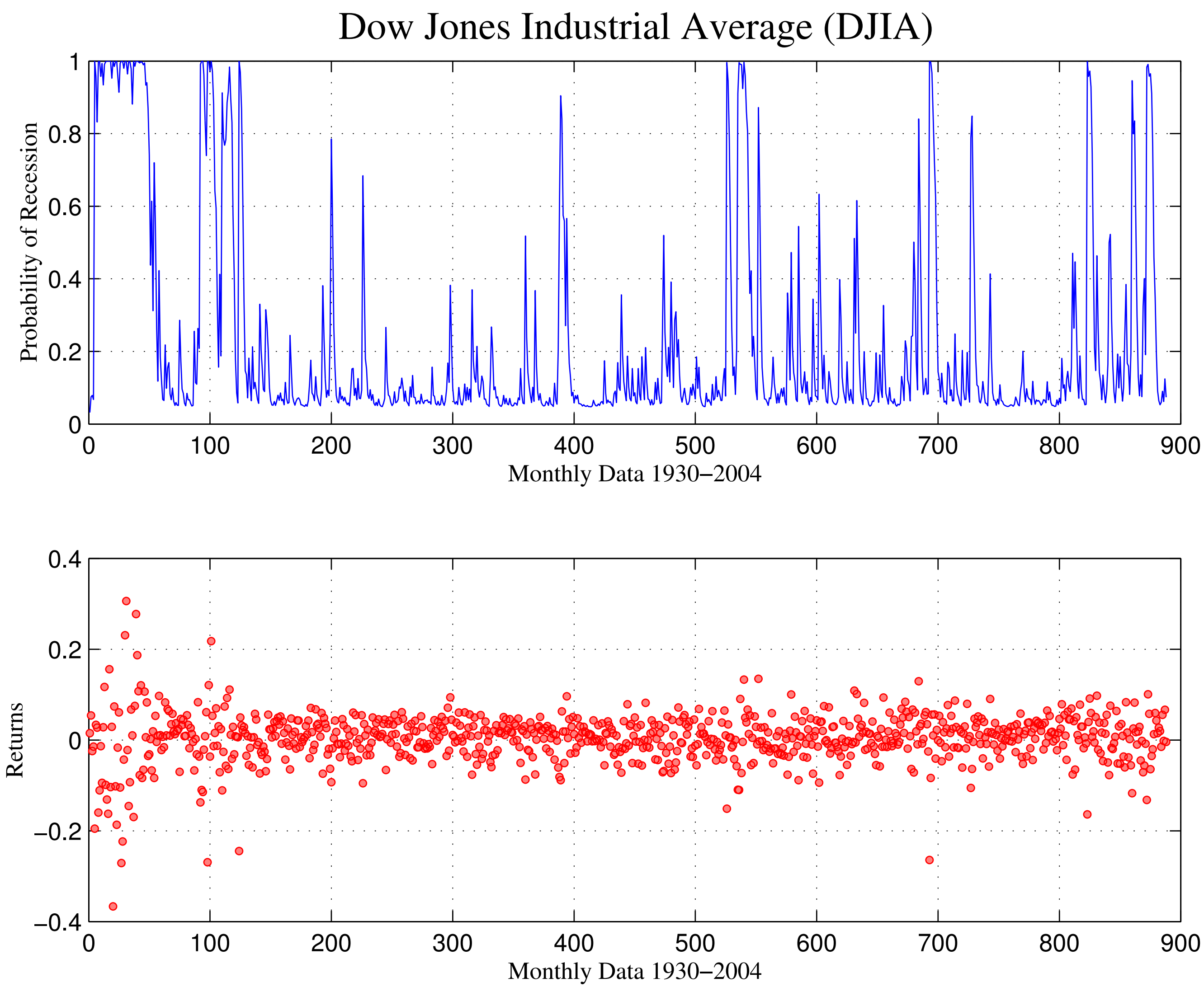
- Using the regime switching approach, a monthly series of the Dow Jones Industrial Average index is estimated
- Two regimes (expansion/recession) are assumed
- The likelihood is maximized numerically, while the standard errors are retrieved by inverting the Hessian matrix
- The recession regime has *higher* volatility and *lower* expected return

An example

The results are summarized in the following table

	Estimate	St Error
μ_1	0.0082	0.0014
μ_2	-0.0178	0.0119
σ_1	0.0380	0.0011
σ_2	0.1237	0.0110
p_{12}	0.0134	0.0052
p_{21}	0.1226	0.0433

An example



Dow Jones Industrial Average (DJIA)
Probability of Recession
0
0.2
0.4
0.6
0.8
1
0
100
200
300
400
500
600
700
800
900
Monthly Data 1930–2004
Returns
0.4
0.2
0
–0.2
–0.4
0
100
200
300
400
500
600
700
800
900
Monthly Data 1930–2004

Questions

1. The c.f. of the normal distribution is given by $\exp\{\mathtt{i}\mu\theta - \frac{1}{2}\sigma^2\theta^2\}$. Compute the density with FFT and compare it to the true density. Also compute option prices using FFT and compare them to the Black-Scholes prices.
2. Gather data on an index and estimate a simple regime switching model. Use two regimes and make inference on the unobserved regime(s). Repeat the same procedure for a three regime specification and compare the two approaches.