Workshop 2

Time Varying Volatility: From GARCH *to* SV

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CCFEA Workshop on Computational Finance

Historical volatility

- Volatility in financial markets varies over time
- This time variation causes fat tails
- The volatility exhibits high autocorrelation
- There is mean reversion present
- There appear profound volatility clusters
- Negative return/volatility correlation supports the leverage effect

Implied volatility

- The IV equates the actual option prices with their Black-Scholes values
- Short at-the-money IV reflects expected volatility
- Aggregate IV such as the VIX index show autoregressive structure
- The term structure of IV shows some long run trend
- The cross section of IV exhibits a smile before 1987 and a smirk after that (crash fears)
- It is challenging to achieve the observed IV from a volatility model

Two approaches

To model the time varying nature of the asset return volatility one has to choose between a GARCH and a SV approach

	GARCH	SV
current vol	known	unknown
conditional vol	computable	unknown
vol randomness	no source	extra source
vol risk price	set internally	set externally
time frame	discrete	continuous
incompleteness	discrete time	extra diffusions
option pricing	very limited	available
param estim	max likelihood	hard

The GARCH model

The GARCH model introduces autoregressive conditional heteroscedasticity in discrete time. In the most popular GARCH (1,1) the conditional mean and variance have the form

$$\begin{aligned} X_t &= \mu + \sigma_t \epsilon_t \\ \sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \gamma \sigma_{t-1}^2 \epsilon_{t-1}^2 \end{aligned}$$

under the constraints $\omega, \beta, \gamma \geq 0$, $\beta, \gamma \leq 1$, $\beta + \gamma < 1$.
The unconditional variance is $\bar{\sigma}^2 = \frac{\omega}{1-\beta-\gamma}$, something which is also used frequently as a constraint.

The GARCH likelihood

The fact that *conditionally* the random variables X_t under GARCH model are normally distributed, allows one to compute the likelihood for a given set of parameters

$$\log \mathscr{L}_t = -\frac{(X_t - \mu)^2}{2\sigma_t^2} - \frac{1}{2}\log \sigma_t^2 - \frac{1}{2}\log 2\pi$$

The econometrician can find the parameters $\vartheta = \{\mu, \omega, \beta\gamma\}$ that maximizes

$$\vartheta^{ML} = \arg\max_{\vartheta} \sum_{t=1}^{T} \log \mathscr{L}_t$$

Estimation example

As an example, the estimation of the monthly Dow Jones Industrial index from 1930 to 2004 gives the estimates

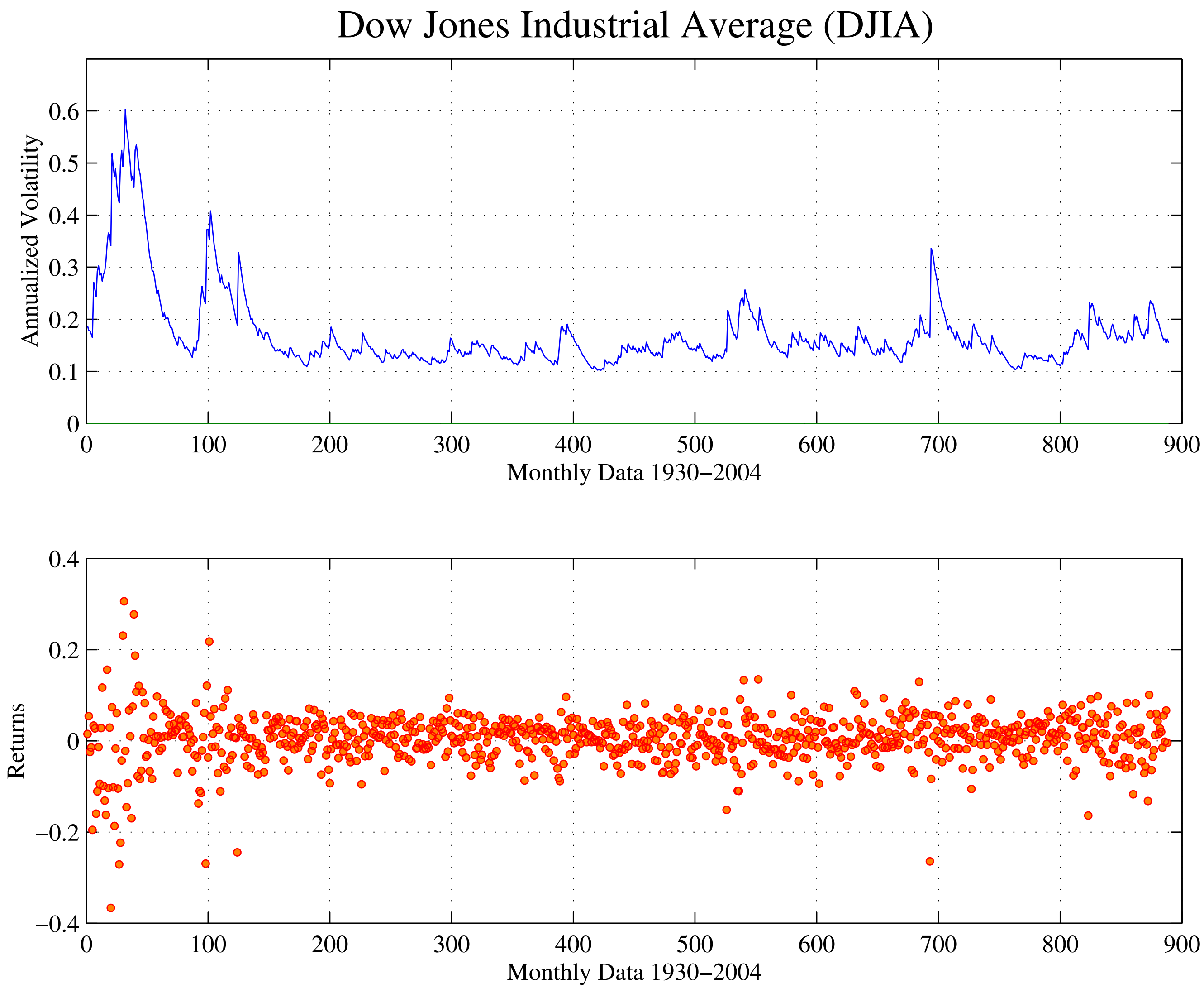
μ	0.0066	(0.0010)
β	0.8745	(0.0131)
γ	0.1013	(0.0101)

The parameter ω is restricted as for the long run variance to match the sample variance $\omega = \bar{\sigma}^2(1 - \beta - \gamma)$.

Estimation example cont'd

- One can observe that the composition of the variance comes roughly 87% from the past variance, 10% from the effect of the new disturbance, and only 3% from the trend towards the long run volatility.
- The sum $\beta + \gamma \approx 0.98$ which implies a very *slow* mean reversion. A sum equal to one would suggest an integrated GARCH or EWMA (exponentially weighted moving average) specification, where the variance is updated as $\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)\sigma_{t-1}^2\epsilon_{t-1}^2$.

Estimation example cont'd'd



GARCH option pricing

- The market is incomplete: There is not a unique way to identify the risk neutral probability measure in discrete time models.
- Replicating portfolios do not exist: The state-space is too dense for the trading time-space.

Approach I

- Assumptions can be made on the utility $U(W_t)$ and on the relationship between W_t and X_t.
- Then option prices can be computed from the Euler equations, e.g. the price of a European call option would be

$$C(t, T, K) = \mathsf{E}\left\{\frac{U_W(W_T)}{U_W(W_t)}(X_T - K)^+\right\}$$

- It is not straightforward neither to specify the appropriate utility nor to compute the expectation in closed form

Approach II

- Suppose that we have the log price following

$$\Delta \log S_t = \mu - \frac{1}{2}\sigma_t^2 + \sigma_t \epsilon_t$$
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma\sigma_{t-1}^2\epsilon_{t-1}^2$$

- We *define* as the risk neutral probability measure the one under which the random variable

$$\epsilon_t^{\mathsf{Q}} = \epsilon_t - \frac{r-\mu}{\sigma_t}$$

is a martingale

Approach II cont'd

- Then under risk neutrality the asset log price follows

$$\begin{aligned}\Delta \log S_t &= r - \frac{1}{2}\sigma_t^2 + \sigma_t \epsilon_t^{\mathsf{Q}} \\ \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma\sigma_{t-1}^2 \left[\epsilon_{t-1}^{\mathsf{Q}} + \frac{r-\mu}{\sigma_t}\right]^2\end{aligned}$$

- Derivative are computed as the expectation under risk neutrality, e.g.

$$C(t,T,K) = e^{-r(T-t)}\mathsf{E}^{\mathsf{Q}}\left\{(S_T - K)^+\right\}$$

Approach II cont'd'd

- An implicit assumption is that the conditional risk neutral density is also normal [or BS holds for $T = 1$].
- The expectation is not generally computable in closed form. Simulation based techniques can be employed.
- The characteristic function is available for a similar class

$$\begin{aligned}\Delta \log S_t &= r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\epsilon_t \\ \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma\epsilon_{t-1}^2\end{aligned}$$

- Then FFT can be used

The SV model

- SV is based in continuous time, although discrete versions are available
- The general form is

$$\begin{aligned} \mathtt{d}\log S_t &= \left(\mu - \frac{1}{2}v_t\right)\mathtt{d}t + \sqrt{v_t}\mathtt{d}W_t \\ \mathtt{d}v_t &= \alpha(v_t)\mathtt{d}t + \beta(v_t)\mathtt{d}Z_t \end{aligned}$$

with some correlation between $\mathsf{dCov}\{W_t, Z_t\} = \rho\mathtt{d}t$

Special SV models

- The square root model of Heston

$$\begin{aligned} \mathtt{d}v_t &= \theta(\bar{v} - v_t)\mathtt{d}t + \phi\sqrt{v_t}\mathtt{d}Z_t, \text{ or} \\ \mathtt{d}v_t &= (\kappa - \theta v_t)\mathtt{d}t + \phi\sqrt{v_t}\mathtt{d}Z_t \end{aligned}$$

- The log-variance model of Hull-White

$$\begin{aligned} \mathtt{d}U_t &= \theta(\log\bar{v} - U_t)\mathtt{d}t + \phi\mathtt{d}Z_t, \text{ with} \\ v_t &= e^{U_t} \end{aligned}$$

SV estimation

- Since in SV models the volatility is unobserved, it is *very* hard to filter the parameter values. People have used:
  - Indirect inference: via EGARCH
  - Simulation based methods: SMM, EMM, SML, MCMC, PF, UPF
  - Approximate likelihood methods: KF, EKF, UKF, MC-ML
- The results are not satisfactory

Option pricing

- Equivalent measures to the the objective one will have measures of the form

$$\frac{\mathrm{dQ}}{\mathrm{dP}} = \exp\left\{\int_t^T \Phi_s \mathrm{d}W_s - \frac{1}{2}\int_t^T \Phi_s^2 \mathrm{d}s + \int_t^T \Psi_s \mathrm{d}Z_s - \frac{1}{2}\int_t^T \Psi_s^2 \mathrm{d}s\right\}$$

- The integrands Φ and Ψ can be any functions on (S_t, v_t).

Option pricing cont'd

- Taking $\Phi = \frac{r-\mu}{\sqrt{v_t}}$, which is the price of risk, is enough to make Q a risk neutral measure.
- Then Ψ is reflecting the *price of volatility risk*
- Under Q the process becomes

$$\begin{aligned} \mathtt{d}\log S_t &= \left(r - \frac{1}{2}v_t\right)\mathtt{d}t + \sqrt{v_t}\mathtt{d}W_t^{\mathsf{Q}} \\ \mathtt{d}v_t &= \alpha^{\mathsf{Q}}(v_t)\mathtt{d}t + \beta(v_t)\mathtt{d}Z_t^{\mathsf{Q}} \end{aligned}$$

with $\alpha^{\mathsf{Q}} = \alpha + \beta\rho\Phi + \beta\sqrt{1-\rho^2}\Psi$. If the latter part is a constant, then $\alpha^{\mathsf{Q}} = \alpha + \zeta$.

Affine models

- Affine models are ones with a linear structure on the drift and the covariance matrix
- The Heston model is an affine one
- For affine models the characteristic function can be computed, for some (like Heston) in closed form
- Then FFT can be used

The calibration

- Instead of estimating the parameters from *historical* asset prices, one can retrieve them from *today's* option prices
- Such parameters are the risk neutral ones and *should* be stationary through time. They are a generalization of the BS implied volatilities
- This calibration can be done by minimizing the distance between theoretical and actual option prices
- One can use these parameters to price more exotic contracts *in a way that is consistent with the observed market prices*

Calibration example

- For example on 17-Sept-1993 the calibrated parameters for the Heston model were

v_0	0.0098
θ	16.7459
$\bar{v}$	0.0123
ϕ	1.1225
ρ	-0.6238

- These parameters make the theoretical volatility surface to resemble the observed one...

The calibrated IV surface